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Spherically symmetric clusters of charged particles

A Banerjee and D Bhattacharya

Department of Physics, Jadavpur University, Calcutta 700032, India

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Abstract. It is shown that the Reissner–Nordström singularity is unattainable in the case of a stationary spherically symmetric cluster of charged particles with a motion analogous to that of an Einstein cluster. It is further shown that photons or neutrinos moving radially outwards inside the system cannot be trapped within the cluster itself.

1. Introduction

Einstein (1939) has shown that a spherically symmetric stationary cluster of many gravitating masses describing circular orbits may be constructed. The particles are assumed to move with perfectly arbitrary phases and orientations under the influence of the gravitational field produced by all of them. An interesting result of this work is that the particles at the boundary in such a case are constrained to move beyond a certain critical distance; this leads one to the conclusion that there is a limit to the concentration, so that the Schwarzschild singularity is unattainable in this case.

The object of this paper is to investigate whether there is any such restriction on the radius of the cluster when the orbiting particles are charged. It is found that there is indeed a limit to the size of the cluster for a given total mass and charge, so that the Reissner–Nordström singularity does not appear at the boundary.

In the second part of the paper (§ 3) it is further shown that neutrinos or photons travelling along the paths of null geodesics in the outward direction inside the distribution must reach the surface in all cases; in other words there is no trapping of such particles moving in the outward direction within the cluster itself. This result is exactly identical to that obtained in the case of uncharged particles (Hogan 1973).

2. The energy–momentum tensor and the field equations

The gravitational field of the cluster has spherical symmetry about a centre and each particle describes its orbit under the influence of this field, the effect of collisions between particles being neglected. Since the particles carry charge, there will be an electric field in the radial direction only.

Now the gravitational field within the spherically symmetric cluster is given by

$$ds^2 = e^{\nu}\, dt^2 - e^{\lambda}\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \tag{1}$$

where λ and ν are functions of r alone and $r \leqslant a$. $r = a$ gives the boundary of the system.

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Since there is only a radial electric field, the only non-vanishing component of $F^{\mu\nu}$ is F^{14}. Maxwell's equations $F^{\mu\nu}_{;\nu} = 4\pi j^{\mu}$ leads us to

$$F^{14} = -A(r)/e^{(\nu+\lambda)/2}r^2 \qquad \text{for } \mu = 1 \tag{2}$$

and

$$j^4 = A'(r)/4\pi\, e^{(\nu+\lambda)/2}r^2 \qquad \text{for } \mu = 4 \tag{3}$$

where $A(r)$ is an arbitrary function of r and j^{μ} is the four-current density. It should be remembered that because of the spherical symmetry the only non-vanishing component of j^{μ} is j^4. $j^1 = j^2 = j^3 = 0$ since the charged particles have no radial motion and rotate perfectly at random. At any point within the cluster they move in tangential directions with the same speed, the same number of particles moving in all directions. The charge density is thus given by

$$\sigma^2 = j^{\mu} j_{\mu},$$

so that

$$4\pi\sigma = \pm A'(r)/e^{\lambda/2}r^2. \tag{4}$$

Therefore, in view of the static spherically symmetric nature of the metric, the matter and the electromagnetic field may be given by (Teixeira and Som 1974)

$$T^{\mu}_{\nu} = \rho_0[0, -\tfrac{1}{2}\alpha^2, -\tfrac{1}{2}\alpha^2, (1+\alpha^2)] \tag{5}$$

and

$$4\pi\tau^1_1 = -4\pi\tau^2_2 = -4\pi\tau^3_3 = 4\pi\tau^4_4 = A^2/2r^4 \tag{6}$$

where T^{μ}_{ν} and τ^{μ}_{ν} represent energy tensors for matter and the electromagnetic field respectively. $\rho_0 \equiv T$ and α is a function of r alone.

One can write the field equations

$$G^0_0 = e^{-\lambda}(r^{-2} - \lambda' r^{-1}) - r^{-2} = -8\pi\rho_0(1+\alpha^2) - (A^2/r^4) \tag{7}$$

$$G^1_1 = e^{-\lambda}(r^{-2} + \nu' r^{-1}) - r^{-2} = -(A^2/r^4) \tag{8}$$

$$G^2_2 = G^3_3 = e^{-\lambda}(2r\nu'' + r\nu'^2 + 2\nu' - r\nu'\lambda' - 2\lambda')/4r = 4\pi\rho_0\alpha^2 + (A^2/r^4) \tag{9}$$

where G^{μ}_{ν} is the Einstein tensor given by

$$G^{\mu}_{\nu} = R^{\mu}_{\nu} - \tfrac{1}{2}\delta^{\mu}_{\nu}R.$$

The Bianchi identity gives

$$\rho_0[\alpha^2 r^{-1} - \tfrac{1}{2}\nu'(1+\alpha^2)] = -(A^2)'/8\pi r^4. \tag{10}$$

Here a prime indicates differentiation with respect to r. Since $A^2(r)$ has the significance of the square of the charge integral within the sphere of radius r (see Bekenstein 1971), $(A^2)' > 0$ for charges of the same sign.

In view of the above fact, the relation (10) shows immediately that $\nu' > 0$ which, combined with the field equation (8), yields the relation

$$e^{\lambda}(1 - A^2/r^2) \geqslant 1. \tag{11}$$

Now for regularity at the origin, as $r \to 0$, $e^{\lambda} \to 1$ and one may further conclude, in view of (4), that $A^2/r^2 \to 0$ as $r \to 0$ for a finite value of the charge density at this point. The relation (8) therefore tells us that $\nu' r$ vanishes at the origin.

Now we can write the Bianchi identity (10) in the form

$$\alpha^2(1-\tfrac{1}{2}\nu' r)=\tfrac{1}{2}\nu' r-((A^2)'/8\pi\rho_0 r^3). \tag{12}$$

One has to conclude that $\frac{1}{2}\nu' r$ cannot be greater than one at the boundary, since in that case it would be exactly equal to unity somewhere within the cluster, and $\frac{1}{2}\nu' r=1$, in view of (12), means that α is of infinite magnitude. $\alpha\to\infty$ again has the implication that the particle reaches the velocity of light at this point; this can be seen from the equations of motion of any of the charged particles of the cluster in the following way.

By putting $\mu=1$ in the equation of motion

$$\rho_0\left[\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2}+\Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\frac{\mathrm{d}x^\beta}{\mathrm{d}s}\right]=-F^\mu_{\ \alpha}j^\alpha \tag{13}$$

for a particle moving in the plane $\theta=\frac{1}{2}\pi$ one obtains

$$\rho_0\,\mathrm{e}^{-\lambda}[r(\mathrm{d}\phi/\mathrm{d}s)^2-\mathrm{e}^{\nu}(\mathrm{d}t/\mathrm{d}s)^2\tfrac{1}{2}\nu']=-(A^2)'\,\mathrm{e}^{-\lambda}/8\pi r^4. \tag{14}$$

Also from (1),

$$\mathrm{e}^{\nu}(\mathrm{d}t/\mathrm{d}s)^2-r^2(\mathrm{d}\phi/\mathrm{d}s)^2=1. \tag{15}$$

Equations (14) and (15) now give

$$\mathrm{e}^{\nu}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2=\frac{1-((A^2)'/8\pi r^3\rho_0)}{1-\frac{1}{2}\nu' r}. \tag{16}$$

Comparing the velocities of a particle and a light ray traversing circular paths in the plane $\theta=\frac{1}{2}\pi$ where $v^2\,\mathrm{e}^{-\nu}$ is equivalent to the ratio of the square of the velocity of a particle to the square of the light velocity (Gilbert 1954) and using (12) and (16) we get the relation

$$v^2\,\mathrm{e}^{-\nu}=\alpha^2/(1+\alpha^2). \tag{18}$$

So when $\alpha\to\infty$, $\alpha^2/(1+\alpha^2)\to 1$ and the velocity of the particle approaches the velocity of light.

So finally we obtain the condition that $\frac{1}{2}\nu' r<1$ at the boundary and from (11) we get the inequality

$$1\leqslant \mathrm{e}^{\lambda(r_0)}(1-Q^2r_0^{-2})<3 \tag{19}$$

where $\lambda(r_0)$ is the value of λ at $r=r_0$ and Q is the total charge content of the cluster. The relation (19) can also be written in a more explicit form utilising the exterior Reissner–Nordström solution as

$$1\geqslant\frac{1-2Mr_0^{-1}+Q^2r_0^{-2}}{1-Q^2r_0^{-2}}>\frac{1}{3}. \tag{20}$$

Now in order that the event horizon appears at the boundary for $M>|Q|$, r_0 must approach $M+(M^2-Q^2)^{1/2}$, which is greater than $|Q|$. Thus $(1-Q^2r_0^{-2})$ remains positive and finite, whereas $(1-2Mr_0^{-1}+Q^2r_0^{-2})$ vanishes, which contradicts (20).

Again for $M^2=Q^2$, the relation (20) reduces to

$$1\geqslant\frac{1-|Q|r_0^{-1}}{1+|Q|r_0^{-1}}>\frac{1}{3}. \tag{21}$$

So the event horizon appears at the boundary $r = r_0$ when r_0 approaches M $(=Q)$. In that case

$$\frac{1-|Q|r_0^{-1}}{1+|Q|r_0^{-1}} \to 0,$$

which is again inconsistent with (21). Thus for fixed M and $|Q|$, one can decrease the dimension of the cluster only up to a certain limit of concentration without the event horizon appearing at the boundary of the cluster at any stage. Equation (20) leads directly to bounds on r_0, that is, on the dimensions of such a cluster, by the relations

$$r_0 \geqslant Q^2/M \tag{22a}$$

and

$$1-3Mr_0^{-1}+2Q^2r_0^{-2}>0. \tag{22b}$$

The second condition in fact yields the relation $r_0^2 > Q^2$ in view of the first.

3. Emission of neutrinos

It is well known that the equations of motion of neutrinos are given by null geodesics and such paths within the cluster under consideration are given by

$$e^{\nu}\dot{t}^2 - e^{\lambda}\dot{r}^2 - r^2\dot{\phi}^2 = 0, \qquad r^2\dot{\phi} = \alpha, \qquad e^{\nu}\dot{t} = \alpha\beta \tag{23}$$

where α and β are constants of motion. The dot represents differentiation with respect to some affine parameter. Thus from (23) one obtains (Hogan 1973)

$$r^{-4}\, e^{(\lambda+\nu)}(dr/d\phi)^2 = (\beta^2 - e^{\nu}r^{-2}). \tag{24}$$

Such paths have apses when $e^{\nu}r^{-2} = \beta^2$ at some r. Again

$$(d/dr)(e^{\nu}r^{-2}) = 2\, e^{\nu}r^{-3}(\tfrac{1}{2}\nu' r - 1). \tag{25}$$

Since it has been argued earlier that within the cluster $\frac{1}{2}\nu' r < 1$ we get $(d/dr)(e^{\nu}r^{-2}) < 0$ and so once the neutrino is emitted from any point within the cluster in the outward direction, it does not turn within the cluster, because with increasing r the quantity $e^{\nu}r^{-2}$ continues to decrease and cannot be equal to β^2 before the neutrino reaches the surface. One may therefore conclude that even when the particles in an Einstein cluster are charged, the outgoing photons or neutrinos cannot be trapped within the cluster.

References

Bekenstein J D 1971 *Phys. Rev.* D **4** 2185
Einstein A 1939 *Ann. Math.* **40** 922
Gilbert C 1954 *Mon. Not. R. Astron. Soc.* **114** 628
Hogan P A 1973 *Proc. R. Irish Acad.* **73** 91
Teixeira A F da F and Som M M 1974 *J. Phys. A: Math., Nucl. Gen.* **7** 838